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Prospect theory and fat tails

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Abstract. A behavioral representative investor who evaluates a single risky asset based on cumulative prospect theory will often induce high kurtosis, negative skewness, and persistent autocorrelation into the distribution of market returns even if the asset payoffs are merely a sequence of independent coin tosses. These findings continue to hold even when the investor is simply loss averse.
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What causes fat tails and extreme events in market returns? One possibility is that the market prices accurately reflect the underlying business risk, and that business risk itself has rare but extreme possibilities. This possibility is the implicit assumption in statistical models of market returns. The business risks are presumed to be reflected in the market and so we study the market process to deduce the distribution of the underlying business risks.

The alternate possibility, and the one I follow here, is that the market process itself may augment the possibility of extreme risks, even when the underlying business risk has no rare but extreme events.

How do we know which possibility is the right one? The second does makes a specific but hard to test prediction: if we could observe two markets on the same asset, one by human traders subject to standard behavioral tendencies and psychological biases, and one by risk neutral robots, the behavioral market would have more extreme events than the risk neutral one. Experimental results do suggest that bubbles and crashes are a product of human trading and can dissipate as experience and group familiarity grows, cf. [3] for a review of 72 such experiments.

The aim of this paper is to see if applying standard behavioral models of investor psychology and decision making to the repeated evaluation of a sequence of binomial gambles generates new extreme events in the market prices that do not occur in the underlying business risk.

Suppose there is a single representative investor trading a single market asset whose fundamental risk is as benign as a coin toss, with no extreme events, and known probabilities. If the investor is risk neutral, the asset will always be worth its expected value, and because the expected value will not change in any extreme way over time, neither will the returns of the market asset. Similarly, if the investor maximizes the expected utility of his total wealth, for standard utility functions, no new extreme events are introduced.

However, research over the past few decades has shown that actual investors appear to be neither risk neutral nor expected utility maximizers. A few stylized facts have emerged: people tend to be loss averse, feeling about twice as much pain from losses as they feel pleasure from gains; people evaluate opportunities based on changes to their wealth, not on the overall levels of their wealth; and people tend to be risk-seeking in the domain of losses, willing to overpay for gambles that might reduce their loss, and risk-averse in the domain of gains, as they are scared of losing what they have earned so far. These three observations form the basis of the cumulative prospect theory of [5].

Another consistent psychological observation of human behavior is mental accounting [4]. Mental accounting recognizes that people tend to view their assets in separate accounts, evaluating salary income differently from bonus income, savings money from vacation money, and so forth. In addition, people do not ignore sunk costs: as [2] has shown, individual investors display a disposition effect, a tendency to sell winning stocks but hold on to losing stock in the hope that they recover their prior losses, even if they wouldn't re-invest in those losing stocks if they were forced to liquidate their positions and realize their losses.

It could be the case that even such a behavioral representative investor would still not generate extreme events; after all, he is merely evaluating risky assets a little differently, and the theories and models of behavioral finance have been put forth and tested to match the psychological realities faced by human traders, not explicitly to model fat tails or extreme events.

But it turns out that with reasonable parameter assumptions, stable fundamental risk indeed gets transformed into market prices with high kurtosis, negative skewness, and persistent autocorrelations, all of the troubling features of real markets.

This approach is offered as a proof of concept that we need not merely pick statistical models that fit the data from the top down but that we can explore human psychology to generate price paths from the bottom up.

I build the model with examples and intuition in Section 1, explore its implications in detail in Section 2, discuss and consider simple loss aversion instead of the entirety of cumulative prospect theory in Section 3, and conclude with directions for future research in Section 4.

1. Model

There is a single risky asset in the market that exists for T periods and pays off a coin toss each period. Each coin toss, g, gives u with probability π and d with probability $1-\pi$. Denote by $g(T)$ the distribution resulting from T independent coin tosses. Any gamble X is a list of payoffs and associated probabilities. Sort these payoffs to express the gamble as:

$$X = \{(x_{-m}, q_{-m}); \ldots; (x_{-1}, q_{-1}); (x_0, q_0); (x_1, q_1); \ldots; (x_n, q_n)\},$$

where $x_i < x_j$ for $i < j$, $x_0 = 0$, and the q_i are the probabilities of having the associated payoff x_i.

There is a single behavioral representative investor who evaluates gambles based on the cumulative prospect theory of Kahneman and Tversky [5]. Specifically, his evaluation of gamble G is $v[G]$ where $v[\cdot]$ is the cumulative prospect theory valuation function:

$$v[X] \equiv \sum_{i=-m}^{n} q_i^* v^*(x_i),$$

where

$$v^*(x) = \begin{cases} x^\alpha & \text{for } x \geqslant 0, \\ -\lambda(-x)^\alpha & \text{for } x < 0, \end{cases}$$

$$q_i^* = \begin{cases} w(q_i + \cdots + q_n) - w(q_{i+1} + \cdots + q_n) \\ \quad \text{for } 0 \leqslant i < n, \\ w(q_{-m} + \cdots + q_i) \\ \quad\quad - w(q_{-m} + \cdots + q_{i-1}) \\ \quad \text{for } -m \leqslant i < 0, \end{cases}$$

and

$$w(q) = \frac{q^\delta}{(q^\delta + (1-q)^\delta)^{1/\delta}}.$$

The parameters estimated by [5] from experimental data are $\alpha = 0.88$, $\lambda = 2.25$, and $\delta = 0.65$.

The cumulative prospect theory value of a gamble takes three steps: first, all of the payoffs are evaluated based on v^*, which incorporates loss aversion through the fact that $\lambda > 1$ and concavity over gains and convexity over losses through the fact that $\alpha > 0$; secondly, the probabilities are adjusted to reflect the propensity of investors to overweigh extreme outcomes; finally, the sum of the product of the translated payoffs and the translated probabilities computes the value of the gamble to a behavioral investor.

Like utility, the prospect theory value of a gamble is used to compare two gambles: the one with the higher value is the presumed choice of the behavioral investor. A sure amount P is a gamble that pays off P with probability one.

At each time $t = 0, \ldots, T-1$, the representative investor holds one unit of the risky asset and determines the market price that makes him indifferent between holding the risky asset or holding cash. The certainty equivalent C of the asset at time $t = 0$ is the solution C_0 to the following equation:

$$v[g(T)] = v[C_0].$$

Consider a numerical example where $T = 10$, $\pi = 0.5$, $u = 300$, and $d = -100$. Then $C_0 = 753.86$. By comparison, the expected value of ten such coin tosses is $E[g(10)] = 1000$. Figure 1 shows the ratio of the certainty equivalent to the expected value for T ranging from 1 to 100. The more coin flips, the closer the price gets to the expected value. Appendix A proves that under loss aversion the limit of the ratio approaches one as the number of coin flips increases to infinity.

At time $t = 1$, suppose that the results of the first coin toss are such that the asset returned $A_1 \in \{u, d\}$, giving the investor an unrealized gain or loss. He evaluates the asset relative to his original entry point so that the new certainty equivalent C_1 is found from:

$$v[g(T-1) + A_1] = v[C_1],$$

and in general the certainty equivalent C_t at time t is found from:

$$v\left[g(T-t) + \sum_{i=1}^{t} A_i\right] = v[C_t],$$

where A_k is the result of the kth coin toss.

It is more convenient to deal with scaled numbers. Define:

$$p_t \equiv \frac{C_t - \sum_{i=1}^{t} A_i}{E[g(T-t)]} = \frac{C_t - \sum_{i=1}^{t} A_i}{(T-t)(\pi u + (1-\pi)d)}.$$

Then p_t is the price of the gamble, because it is the excess of the certainty equivalent relative to the investor's actual gains and losses to date, expressed as a portion of the expected value of the remaining gamble. The intuition for the numerator is that the investor could in principle choose between continuing to invest in the risky asset or realizing his gains and losses to date and holding cash. The certainty equivalent C_t expressed how much the investor was willing to pay relative to zero to stay in the risky asset; the scaled excess price p_t represents the more interesting number of how much the investor is willing to pay relative to what he has already gained or lost so far, scaled as a portion of the remaining expected value to make the numbers more comparable across different times t.

Note that the initial certainty equivalent equals the initial price, $C_0 = p_0$, so an alternate interpretation of Fig. 1 is that the price is always below the expected value.

Our particular numerical example is useful for two reasons: one, the expected value of a single coin toss is exactly 100, making scaling easy, and two, the prospect theory value of any coin tosses plus any amount A of accumulated unrealized gains and losses will always exceed the prospect theory value of holding A in cash, as we can see in Fig. 2 and as we can prove for the special case of loss aversion in Appendix B, meaning that

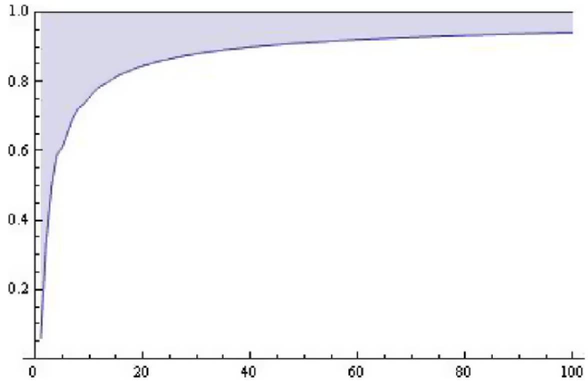

Fig. 1. Coin toss prospect theory valuation. Define the price of a sequence of T fair coin flips paying off 300 on heads and -100 on tails as the certainty equivalent under cumulative prospect theory valuation. This figure shows the ratio of the price to the expected value of the gamble for T ranging from 1 to 100. As the number of coin flips increases, the prospect theory price approaches but never reaches the expected value.

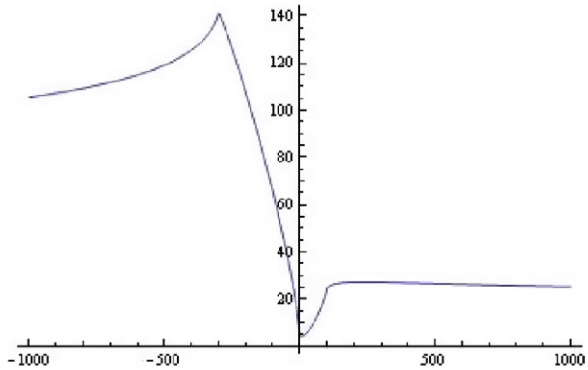

Fig. 2. The graph above is the difference between (a) the prospect theory value of a fair coin toss paying either 300 or -100 plus accumulate profits or losses ranging from -1000 to 1000, and (b) the prospect theory value of the accumulated profits or losses by themselves. Specifically, it is a plot of $v[g+x] - v[x]$, where v is the cumulative prospect theory valuation function, g is the fair coin toss, and x varies along the x-axis from -1000 to 1000.

the investor will always choose the gamble over holding cash, thus assuring positive prices p_t. However, the particular prices at which he is indifferent do change, and it is the distribution of these prices that we wish to explore.

We can solve for the distribution of possible prices $p_t(n)$ where $n \leqslant t$ is the number of heads that have occurred to date. Hence the distribution of p_t is:

$$p_t \sim \left\{ p_t(n) \text{ with probability } \binom{t}{n} \pi^n (1-\pi)^{t-n} \right\},$$

where $p_t(n)$ is such that:

$$v[g(T-t)+nu+(t-n)d] = v[C_t(n)]$$

and as above:

$$p_t(n) = \frac{C_t(n)-(nu+(t-n)d)}{(T-t)(\pi u+(1-\pi)d)}.$$

For the same numerical example, $p_0 = 0.75$ and the distribution of p_1 is:

$$p_1 = \begin{cases} 0.76 & \text{if } A_1 = u, \text{ with probability } p, \\ 0.72 & \text{if } A_1 = d, \text{ with probability } 1-p. \end{cases}$$

In other words, when the first coin toss is up, the price of the asset rises, and when it is down, the price falls, even though the value of the remaining nine coin tosses is independent of that first toss, and even though investors ought to ignore sunk costs by traditional economic reasoning.

This property of behavioral investors to incorporate prior gains and losses into evaluations of future prospects may be part of the explanation for the excess volatility puzzle, or the finding that the stock market tends to move around too much, relative to the volatility of the underlying earnings. As companies report relatively random earnings, investors appear to overreact and cause an even greater price drop, but as Barberis et al. [1] point out, the reason may be that behavioral investors have changing levels of loss aversion resulting from the gains or losses generated by previous market moves. The same effect happens in our model here.

2. Results

Continuing the numerical example from the model, Fig. 3 plots the histogram of the prices across time. Each white label is the digit corresponding to the time t for which the histogram is plotted. Figure 4 plots the individual price histograms for $t = 6, \ldots, 9$. As t increases, the histogram spreads out, and at least visually is far from normal.

Figure 5 plots all of the $2^T = 1024$ possible price paths. All of the paths start from $p_0 = 0.75$ and many

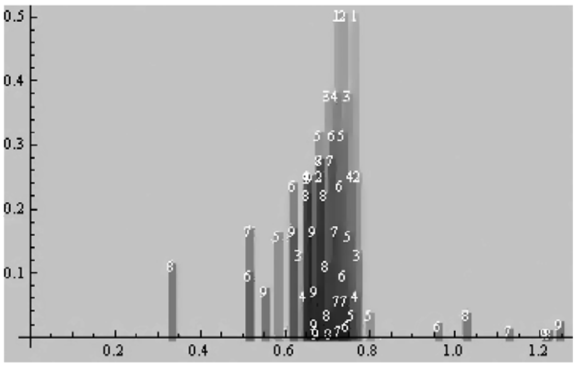

Fig. 3. Histograms of prices. For $T = 10$, $\pi = 0.5$, $u = 300$ and $d = -100$, the graph above plots the histograms of the implied prices of the behavioral representative investor after $t = 1, \ldots, 9$ coin tosses. The bars are labelled with t, so for example the highest possible price of 1.25 occurs with 1.8% probability when $t = 9$.

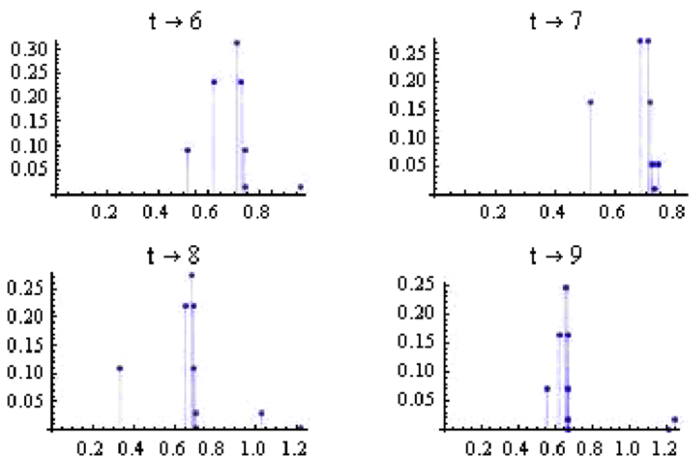

Fig. 4. Final histograms of prices. For $T = 10$, $\pi = 0.5$, $u = 300$ and $d = -100$, the graph above plots the histograms of the implied prices of the behavioral representative investor after $t = 6, \ldots, 9$ coin tosses.

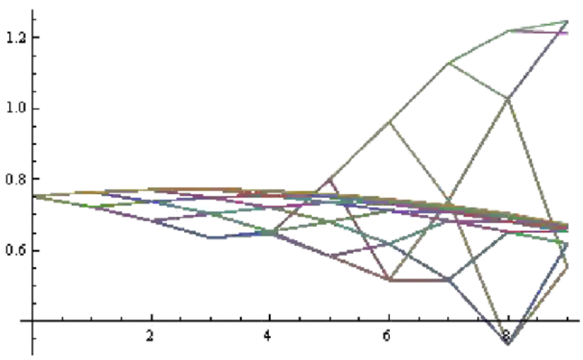

Fig. 5. All price paths. Each line in the graph below represents one possible path of prices implied by the behavioral representative investor following $t = 0, \ldots, 9$ of the $T = 10$ coin tosses that return $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$.

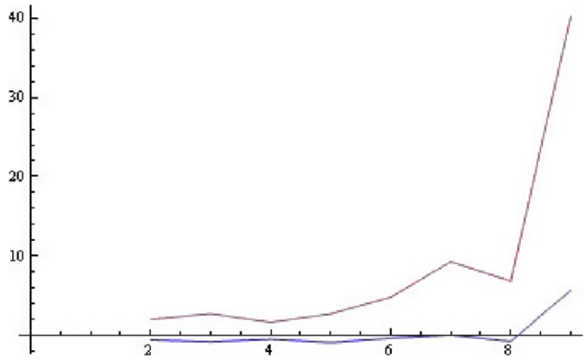

Fig. 6. Kurtosis and skewness. The top line shows the kurtosis of the implied price distributions of the behavioral representative investor after t out of $T = 10$ coin tosses paying out $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$, and the bottom line shows the corresponding skewness.

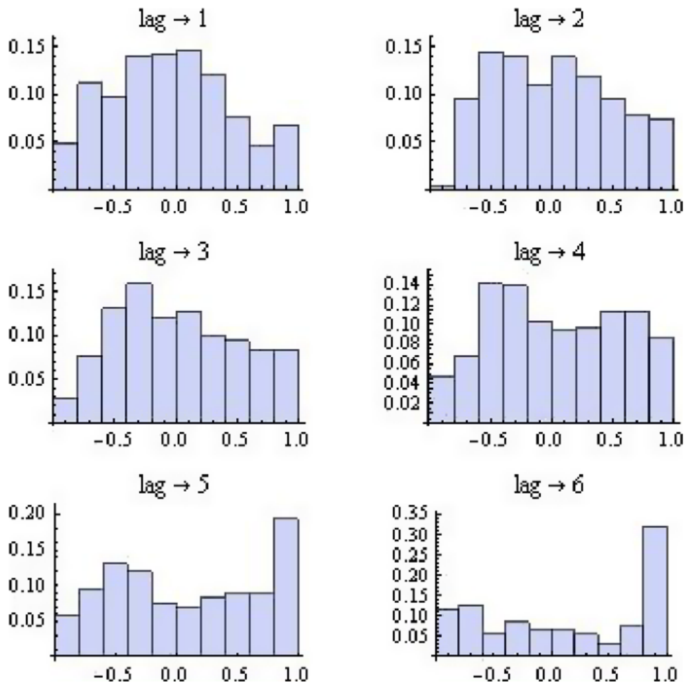

Fig. 7. Autocorrelations. The six graphs above show the histogram of autocorrelations for given lags of returns calculated from the price paths implied by a behavioral representative investor evaluating $T = 10$ coin flips that result in $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$.

of them follow a smooth arc, but several extremes paths are also generated.

We can compute the skewness and kurtosis of the implied distributions as a function of the time t. These are shown in Fig. 6. The kurtosis exceeds three for all $t > 5$, reaching a maximum near 40 at $t = 9$, and the skewness is nearly always negative, except for $t = 9$.

We can also compute the autocorrelations of each path: given a particular price path, we calculate overlapping returns of lag l and compute the correlation between successive such returns. Figure 7 shows the histogram of these autocorrelations across all possible paths. Virtually any autocorrelation is possible, though as the lag increases, a correlation near one emerges as the mode.

3. Discussion

Which of the assumptions of cumulative prospect theory are necessary to generate these results? We can reproduce the results for different values of the cumulative prospect theory parameters. In particular, if the probability weighting parameter of cumulative prospect theory, δ, is set equal to one, then the probabilities are unadjusted, and if the curvature parameter α is also set equal to one, then the prospect theory valuation of a gamble reduces to a straightforward expected value where losses are multiplied by $\lambda = 2.25$.

In this limited model, without risk aversion over gains or risk seeking over losses, and without overweighting the likelihood of extreme events, the same results continue to hold. In other words, it is just the loss aversion and the mental accounting that create extreme events.

Figure 8 plots all of the possible price paths implied by a loss averse investor. As before, the possible prices spread out widely.

Figure 9 shows the skewness and kurtosis of the resulting price distributions. The effects are even more pronounced. The kurtosis exceeds three for all $t > 2$, and the skewness is consistently negative.

Figure 10 shows the lagged autocorrelations of the resulting price series. As before, the possible correlation can be quite high with significant probability.

4. Conclusion

We have seen how simple loss aversion can result in extreme distributions even when the underlying business risk has no extremes. In general, the results hold under cumulative prospect theory, though the minimum required assumptions seem to be only loss aversion, experiencing losses as about twice as painful as gains are pleasant, and mental accounting, incorporating the previous gains and losses on an asset with its future values when evaluating it.

These two assumptions – loss aversion and mental accounting – are among the most well-documented in the behavioral finance literature and the most stable across both individual and institutional investors.

The fact that they also generate extreme market price distributions may suggest that it is the activity of the investors that is causing the extreme events, and not the underlying business risk.

Future research could replace the discrete binomial distribution with a continuous normal or other distribution. We could consider multiple risky assets, or allow for other investors, including the possibility of arbitrageurs and of overlapping generations where new investors enter the market with no accumulated profits or losses.

Appendix: Proofs

Assuming only loss aversion, so that the probability weighting parameter δ and the curvature parameter α of cumulative prospect theory are set equal to one, then the investor with loss aversion parameter $\lambda = 2.25$ evaluates gambles

$$X = \{(x_{-m}, q_{-m}); \ldots; (x_{-1}, q_{-1}); (x_0, q_0); (x_1, q_1); \ldots; (x_n, q_n)\}$$

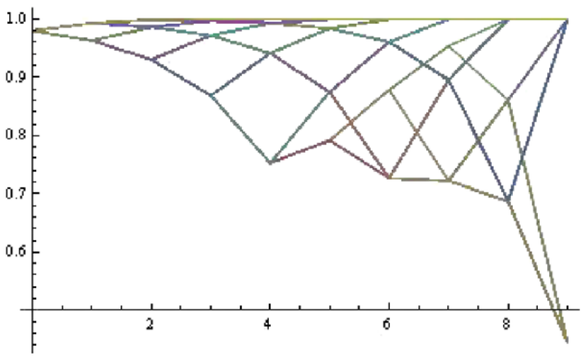

Fig. 8. All price paths for loss aversion. Each line in the graph above represents one possible path of prices implied by the loss averse representative investor following $t = 0, \ldots, 9$ of the $T = 10$ coin tosses that return $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$.

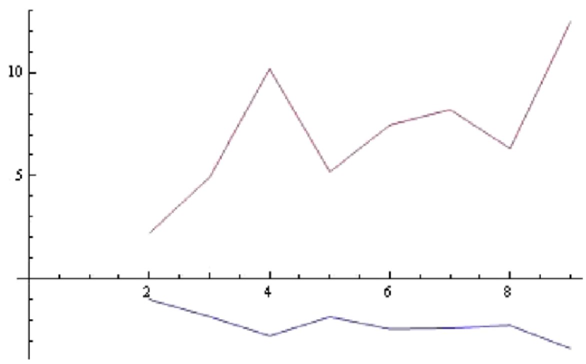

Fig. 9. Kurtosis and skewness under loss aversion. The top line shows the kurtosis of the implied price distributions of the loss averse representative investor after t out of $T = 10$ coin tosses paying out $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$, and the bottom line shows the corresponding skewness.

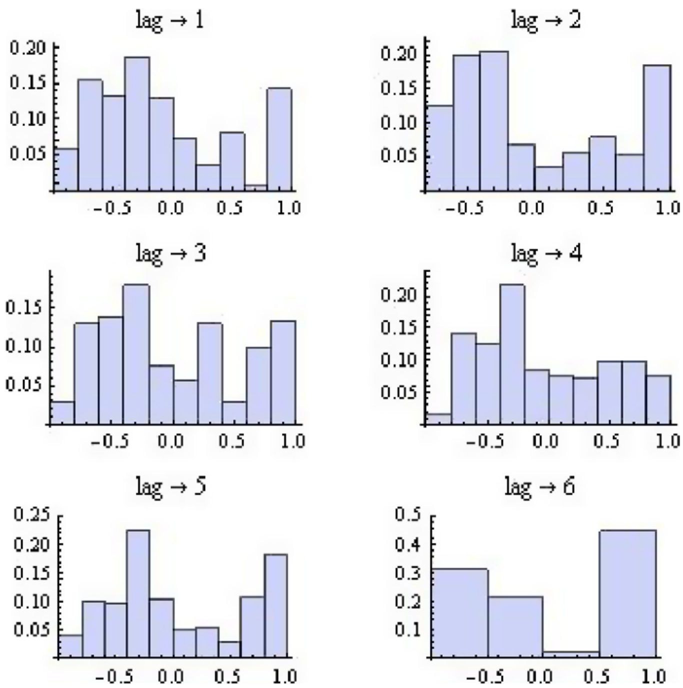

Fig. 10. Autocorrelations under loss aversion. The six graphs above show the histogram of autocorrelations for given lags of returns calculated from the price paths implied by a loss averse representative investor evaluating $T = 10$ coin flips that result in $u = 300$ or $d = -100$ with equal probability $\pi = 0.5$.

with the simpler function:

$$\begin{aligned} v[X] &= \sum_{i=-m}^{-1} \lambda x_i q_i + \sum_{i=0}^{n} x_i q_i \\ &= \sum_{i=-m}^{n} x_i q_i + (\lambda - 1) \sum_{i=-m}^{-1} x_i q_i \\ &= E[X] + 1.25 \sum_{i=-m}^{-1} x_i q_i. \end{aligned}$$

A sequence of T fair coin flips paying off u with probability π and d with probability $1 - \pi$ is the gamble $g(T)$:

$$g(T) = \left\{ \left(ku + (T-k)d, \binom{T}{k} \pi^k (1-\pi)^{T-k} \right) \text{ for } k = 0, \ldots, T \right\}.$$

A. Proof of convergence

In this special case of loss aversion and for our numerical example where $u = 300$, $d = -100$, and $\pi = 0.5$, we can prove that the certainty equivalent of the loss aversion value of the gamble approaches the gamble's expected value in the limit.

Theorem 1. If $\alpha = \delta = 1$, $u = 300$, $d = -100$, and $\pi = 0.5$, then $\lim_{t\to\infty} C_T = E[g(T)]$ where C_T is the certainty equivalent given by $v[g(T)] = v[C_T]$.

Proof. For T coin tosses, we can solve for the minimum number of heads k^* that guarantee a positive outcome:

$$ku + (T-k)d > 0,$$

which implies:

$$k > \frac{T}{4}$$

for our particular u and d. Then the loss aversion value equals the expected value plus 1.25 times the probability-weighted sum of the negative payoffs, or rearranging terms:

$$\begin{aligned}
&E[g(T)] - v[g(T)] \\
&= -\frac{1.25}{2^T}\sum_{k=0}^{T/4}\binom{T}{k}\big(300k - (T-k)100\big) \\
&< \frac{1.25}{2^T}\left(\frac{T}{4}+1\right)\binom{T}{T/4}100T \\
&< \frac{125T^2\binom{T}{T/4}}{2^T}
\end{aligned}$$

and therefore the risk premium approaches zero as T approaches infinity because:

$$\lim_{T\to\infty}\frac{T^2\binom{T}{T/4}}{2^T} = 0.$$

Thus we have shown that $v[g(T)]$ approaches $E[g(T)]$ as T tends to infinity. The certainty equivalent C_T is defined as

$$v[g(T)] = v(C_T) = C_T$$

because C_T is always positive. Therefore the certainty equivalent C_T approaches the expected value $E[g(T)]$ as T approaches infinity.

B. Proof of positivity

In this special case and for our numerical example, we can prove that a loss averse investor will always choose the gamble relative to any starting point.

Theorem 2. If $\alpha = \delta = 1$, $u = 300$, $d = -100$ and $\pi = 0.5$, then $v[g(1) + x] > v[x]$ for all x.

Proof. Consider first the case $x > 0$. Then $v[x] = x$. Call n^* the critical value of the number n of heads such that the payoff from the gamble for $n > n^*$ always exceeds or equals x and the payoff from the gamble for $n \leqslant n^*$ is always less than x. Then we can evaluate:

$$v[g(T) + x] - x$$

as:

$$\begin{aligned}
&\sum_{n=n^*}^{T}\binom{T}{n}\pi^n(1-\pi)^{T-n}\big(nu + (T-n)d + x\big) \\
&\quad + \lambda\sum_{n=0}^{n^*}\binom{T}{n}\pi^n(1-\pi)^{T-n} \\
&\quad \times (nu + (T-n)d + x) - x \\
&= \sum_{n=n^*}^{T}\binom{T}{n}\pi^n(1-\pi)^{T-n}\big(nu + (T-n)d\big) \\
&\quad + \lambda\sum_{n=0}^{n^*}\binom{T}{n}\pi^n(1-\pi)^{T-n} \\
&\quad \times (nu + (T-n)d) \\
&\quad + x\Bigg[-1 + \sum_{n=0}^{T}\binom{T}{n}\pi^n(1-\pi)^{T-n} \\
&\qquad + (\lambda - 1)\sum_{n=0}^{n^*}\binom{T}{n}\pi^n(1-\pi)^{T-n}\Bigg] \\
&= v[g(T)] \\
&\quad + x(\lambda - 1)\sum_{n=0}^{n^*}\binom{T}{n}\pi^n(1-\pi)^{T-n}.
\end{aligned}$$

We have seen from the earlier proof that $v[g(T)]$ is positive for our numerical example. We have assumed x is positive. We know that $\lambda - 1 = 1.25$ is positive. And the final term is just a sum of positive probabilities. Therefore the entire sum is positive, and therefore $v[g(T) + x] - v[x] > 0$ for $x > 0$ and for any T, in particular for $T = 1$.

Now consider the case $x < 0$. Let $y = -x$ be its absolute value so that $y > 0$. Then $v[x] = v[-y] = -\lambda y$ and, defining n^* as above, we can evaluate

$$\begin{aligned}
v[g(T) + x] - v[x] &= v[g(T) - y] - (-\lambda y) \\
&= v[g(T) - y] + \lambda y
\end{aligned}$$

as:

$$\begin{aligned}
&\sum_{n=n^*}^{T}\binom{T}{n}\pi^n(1-\pi)^{T-n}\big(nu + (T-n)d - y\big) \\
&\quad + \lambda\sum_{n=0}^{n^*}\binom{T}{n}\pi^n(1-\pi)^{T-n}
\end{aligned}$$

$$\times\,(nu + (T-n)d - y) + \lambda y$$

$$= \sum_{n=n^*}^{T} \binom{T}{n} \pi^n (1-\pi)^{T-n} \big(nu + (T-n)d\big)$$

$$+ \lambda \sum_{n=0}^{n^*} \binom{T}{n} \pi^n (1-\pi)^{T-n}$$

$$\times\,(nu + (T-n)d)$$

$$+ y\Bigg[\lambda - \sum_{n=0}^{T} \binom{T}{n} \pi^n (1-\pi)^{T-n}$$

$$- (\lambda - 1) \sum_{n=0}^{n^*} \binom{T}{n} \pi^n (1-\pi)^{T-n}\Bigg]$$

$$= v[g(T)]$$

$$+ y(\lambda - 1)\Bigg[1 + \sum_{n=0}^{n^*} \binom{T}{n} \pi^n (1-\pi)^{T-n}\Bigg].$$

As before, we have seen from the earlier proof that $v[g(T)]$ is positive for our numerical example. We have assumed $x = -y$ is negative, so y is positive. We know that $\lambda - 1 = 1.25$ is positive. And the final term is just one plus a sum of positive probabilities. Therefore the entire sum is positive, and therefore $v[g(T) + x] - v[x] > 0$ for $x < 0$.

Therefore we have shown that a loss averse investor will always choose the gamble of T coin tosses per our numerical example for any starting value and any number T.

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